Black Hole Thermodynamics and Riemann Surfaces *

Kirill Krasnov†

Albert Einstein Institute, Golm/Potsdam, 14476, Germany
and
School of Mathematical Sciences, University of Nottingham
University Park, Nottingham, NG7 2RD, UK

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Abstract

We use the analytic continuation procedure proposed in our earlier works to study the thermodynamics of black holes in 2+1 dimensions. A general black hole in 2+1 dimensions has g handles hidden behind h horizons. The result of the analytic continuation is a hyperbolic 3-manifold having the topology of a handlebody. The boundary of this handlebody is a compact Riemann surface of genus $G = 2g + h - 1$. Conformal moduli of this surface encode in a simple way the physical characteristics of the black hole. The moduli space of black holes of a given type (g, h) is then the Schottky space at genus G. The (logarithm of the) thermodynamic partition function of the hole is the Kähler potential for the Weil-Peterson metric on the Schottky space. Bekenstein bound on the black hole entropy leads us to conjecture a new strong bound on this Kähler potential.

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†krasnov@aei.mpg.de

1 Introduction

In this paper we use the analytic continuation procedure proposed in [1, 2] to study the thermodynamics of black hole (BH) solutions of 2+1 gravity with negative cosmological constant. The simplest such black hole was described by Banados, Teitelboim, Zanelli [3]. Later black holes with non-trivial internal topology were discovered [4].

The standard strategy for studying BH thermodynamics is to analytically continue the hole spacetime. The (exponential of the) classical Einstein-Hilbert action evaluated on the resulting Euclidean metric is then the BH partition function. The later can be used to obtain the BH entropy according to the usual thermodynamic formulas. Let us remind the reader how this works for the usual Schwarzschild BH in 3+1 dimensions, see [5]. The Lorentzian signature metric is given by:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2. \tag{1.1}$$

Here M is the BH mass, G is the Newton's constant, $d\Omega^2$ is the line element on the unit sphere, and t, r are the time and radial coordinates correspondingly. The horizon is located at $r = r_+ = 2MG$. To analytically continue the BH spacetime one sends $t \to -i\tau$. The imaginary time coordinate τ must be periodic with period $\beta = 1/T$, where T is the temperature. The Euclidean metric one gets is:

$$ds^2 = \left(1 - \frac{2GM}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2. \tag{1.2}$$

The r, τ part of the metric describes a 2-dimensional "plane" with a conical singularity at the origin $r = r_+$. The period of τ must be chosen in such a way that there is no conical singularity. For a metric of the form:

$$ds^2 = f(r) d\tau^2 + \frac{dr^2}{f(r)}, \qquad f(r_+) = 0 \tag{1.3}$$

the condition that there is no conical singularity is that the circumference $\beta\sqrt{f'(r_+)\epsilon}$ of the circle $r = r_+ + \epsilon$ is equal to 2π times the proper distance $2\sqrt{\epsilon}/\sqrt{f'(r_+)}$ from the origin to the point $r = r_+ + \epsilon$. This gives for the period:

$$\beta = \frac{4\pi}{f'(r_+)}. \tag{1.4}$$

For Schwarzschild BH $f'(r_+) = 1/r_+$ and the temperature one gets from (1.4) is the famous Hawking temperature:

$$T_H = \frac{1}{8\pi GM}. \tag{1.5}$$

To obtain the BH partition function one must evaluate the Einstein-Hilbert action:

$$I[g] = -\frac{1}{16\pi G} \int_{\mathcal{M}} R - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} (K - K_0) \tag{1.6}$$

on the metric (1.2). The first term is identically zero on shell. In the second term K is the trace of the extrinsic curvature of the boundary, and K_0 is the trace of the extrinsic curvature of the boundary embedded in flat space. We have

$$K - K_0 = -\frac{1}{2} f'/f, \tag{1.7}$$

where $f = f(r)$ is as in (1.3). Then, using $f' = r_+/r^2$, computing the integral over the boundary and sending r to infinity we get:

$$I[g_{\rm cl}] = \frac{2\pi r_+ \beta}{8\pi G} = \frac{4\pi r_+^2}{4G}. \tag{1.8}$$

The BH partition function is then $\ln Z = -I[g_{\rm cl}]$.

The standard thermodynamic relations tell us that the expectation value of the energy in the system is given by:

$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta}. \tag{1.9}$$

It is easy to check that $\langle E \rangle = M$, as expected. The entropy is given by:

$$S = -\beta \frac{\partial \ln Z}{\partial \beta} + \ln Z. \tag{1.10}$$

The first term here equals twice the quantity (1.8), while the second term is minus (1.8). Therefore, we get for the entropy:

$$S = \frac{4\pi r_+^2}{4G} = \frac{A}{4G}, \tag{1.11}$$

where A is the horizon area. This is the famous Bekenstein-Hawking entropy.

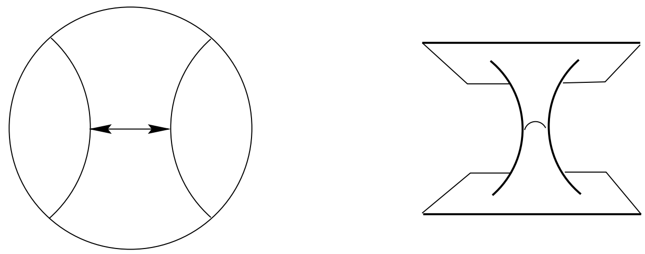
We are going to apply the same strategy to study the thermodynamics of 2+1 dimensional black holes. That is, we are going to analytically continue the BH spacetimes to obtain certain spaces of Euclidean signature. To get the BH partition function we evaluate the gravity action on these spaces. The BH entropy is then obtained according to the standard thermodynamic formulas.

The plan of the paper is as follows. In the next section we shall remind the reader some facts about BH in 2+1 dimensions. In section 3 we describe how to analytically continue the BH spacetimes. The BH partition function is discussed in section 4. Finally, we study the BH thermodynamics in 5. We conclude with a short summary.

2 Black Holes in 2+1 dimensions

The material reviewed in this section is from [4]. The references for the rotating case are [6, 7].

Black holes in 2+1 dimensions can have non-trivial internal topology. A general BH has g handles hidden behind h horizons. Unlike the case with higher-dimensional black holes, BH's in 2+1 cannot have more than one horizon per asymptotic region. Thus, the number of horizons is also the number of asymptotic regions.



(a) (b)

Figure 1: BTZ black hole: the geometry of the time symmetry surface.

As the gravity theory in 2+1 dimensions does not have local degrees of freedom, solutions of vacuum Einstein equations are spacetimes of constant curvature. The case relevant to us here is that of negative cosmological constant. Only in this case there are BH's in the theory. Thus, we have to consider the spacetimes of constant negative curvature. They are all locally indistinguishable from the maximally symmetric spacetime AdS_3. All (complete) spacetimes are therefore obtainable by identifications of points in AdS acting by transformations from some discrete group Γ, subgroup of the group of isometries.

Let us briefly remind the reader some basic facts about the Lorentzian AdS_3. The spacetime is best viewed as the interior of an infinite cylinder. The cylinder itself is the conformal boundary $\mathcal{I}$ of the spacetime. It is timelike, unlike the null conformal boundary of an asymptotically flat spacetime. All light rays propagating inside AdS start and end on $\mathcal{I}$. In this picture the constant time slices are copies of the Poincare (unit) disk. The unit disk is isometric to the upper half plane $\mathbf{U}$; we shall use both models. The isometry group of the Lorentzian signature AdS_3 is $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. The spacetimes itself can be viewed as the $\mathrm{SL}(2,\mathbb{R})$ group manifold so that a point $x \in \mathrm{AdS}_3$ is represented by a matrix $\mathbf{x} \in \mathrm{SL}(2,\mathbb{R})$. The isometry group acts by the left and right action: $\mathbf{x} \to g\mathbf{x}h^{-1}, g, h \in \mathrm{SL}(2,\mathbb{R})$.

Let us now return to the BH spacetimes. It is easiest to describe non-rotating BH's. In this case there is a plane of time symmetry X in the spacetime. In order to have such a plane in the quotient spacetime $M = \mathrm{AdS}_3/\Gamma$, the discrete group Γ that one uses to identify points must be such that its action fixes the $t = 0$ plane in AdS. We use such a parameterization of the $\mathrm{SL}(2,\mathbb{R})$ group manifold that the subgroup of isometries that fixes the $t = 0$ plane is given by: $\mathbf{x} \to g\mathbf{x}g^T, g \in \mathrm{SL}(2,\mathbb{R})$. Note that there is another "diagonal" action of the $\mathrm{SL}(2,\mathbb{R})$ given by: $\mathbf{x} \to g\mathbf{x}g^{-1}$; such transformations fix the origin of AdS. Transformations that fix the $t = 0$ plane of AdS act on this plane. The geometry of the time symmetry plane X of the quotient spacetime is therefore that of $\mathbf{U}/\Gamma$, where $\mathbf{U}$ is the hyperbolic plane. Thus, X is a Riemann surface (with holes, see below) uniformized by $\mathbf{U}$. Once the geometry of the time symmetry plane is understood one just "evolves" the identifications in time to obtain the BH spacetime, see [4].

Let us see how this works on examples. Consider first the case of the non-rotating BTZ BH. The corresponding discrete group is generated by a single hyperbolic element. Its action on the $t = 0$ plane can be understood by finding the so-called fundamental region. The fundamental region D on the hyperbolic plane $\mathbf{U}$ for group Γ is such that any point on $\mathbf{U}$ can be obtained as an image

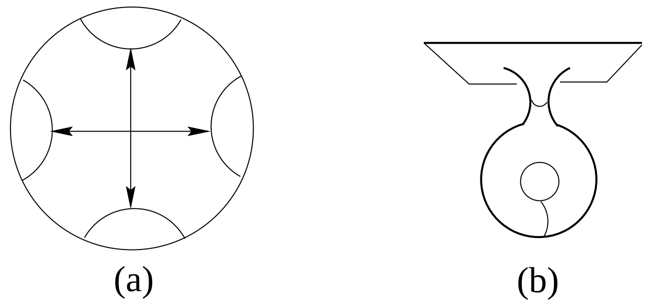


Figure 2: Initial slice geometry of the single asymptotic region black hole with a torus wormhole inside the horizon

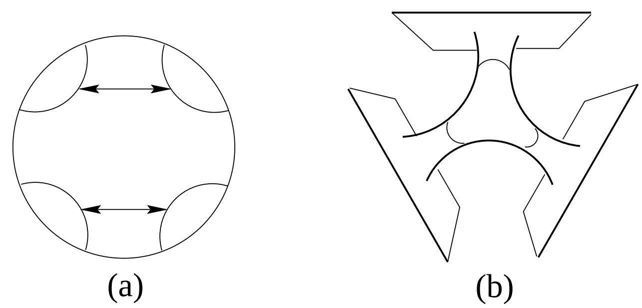


Figure 3: Initial slice geometry of the three asymptotic region black hole

of a point in D under a transformation from Γ, and such that no two points of D (except on its boundary) are related. In the case of Γ generated by a single element γ the fundamental region is that between two geodesics on $\mathbf{U}$ mapped into one another by the generator γ, see Fig. 1(a). It is clear that the quotient space has the topology of the $S^1 \times \mathbb{R}$ wormhole with two asymptotic regions, each having the topology of S^1, see Fig. 1(b). It can also be described as the geometry of a sphere with two holes. On Fig. 1(a) the BTZ angular coordinate runs from one geodesics to the other. The distance between the two geodesics measured along their common normal is precisely the horizon circumference. Evolving the identifications in time one obtains a spacetime of a BH with two asymptotic regions. There are two horizons, they both intersect the time symmetry plane along the minimal length geodesic shown in Fig. 1(b). See [4] for more details on the spacetime picture.

Let us now consider more complicated initial slice geometries. We now consider the group Γ to be generated by two hyperbolic elements. For example, let the fundamental region be the part of the unit disk between four geodesics, as in Fig. 2(a). Let us identify these geodesics cross-wise. It is straightforward to show that the resulting geometry has only one asymptotic region, consisting of all four parts of the infinity of the fundamental region. With little more effort one can convince oneself that the resulting geometry is one asymptotic region "glued" to a torus, see Fig. 2(b). The spacetime obtained by evolving this geometry is a single asymptotic region black hole, but the topology *inside* the event horizon is now that of a torus. See [4] for more details on this spacetime.

A group generated by two elements can also be used to obtain a three asymptotic region black hole [4]. The fundamental region on the $t = 0$ plane is again the region bounded by four geodesics.

They are, however, now identified side-wise, see Fig. 3(a). One can clearly see that the initial slice geometry has three asymptotic regions, as in Fig. 3(b). Evolving this, one gets a spacetime with three asymptotic regions and corresponding event horizons. See [4] for more details.

Taking the group Γ to be more complicated one constructs a large class of spacetimes. In particular, one can have a single asymptotic region black hole with an arbitrary Riemann surface inside the horizon. More generally, one can have a black hole with any number h of asymptotic regions (horizons), and with any number g of handles hidden behind the horizon(s). We shall refer to such a general BH as a hole of type (g, h). The geometry of the time symmetry plane is that of a genus g surface with h holes. The holes are in correspondence with asymptotic regions (horizons). We note that for $g = 0$ the minimal number of holes is 2, which corresponds to the BTZ BH. So we have: $h \geq 2, g = 0$. For $g \geq 1, h \geq 1$.

Let us now consider general, rotating black holes. The idea of [6] was to unravel the spacetime structure by considering the action of the discrete group at the boundary cylinder, instead of thinking about the action at the $t = 0$ plane that is no longer preserved by identifications. One gets a rather effective description, which, for example, allows one to calculate horizon angular velocities in terms of traces of group elements generating identifications. The following formula was obtained in [2] using methods of [6]. To get the angular velocity of a horizon, one must find a group element that generates isometries of the corresponding asymptotic region. The angular velocity is then obtained as:

$$\Omega = \frac{\mathrm{Arccosh}\left(\frac{1}{2}\mathrm{Tr}\,\gamma^L\right) - \mathrm{Arccosh}\left(\frac{1}{2}\mathrm{Tr}\,\gamma^R\right)}{\mathrm{Arccosh}\left(\frac{1}{2}\mathrm{Tr}\,\gamma^L\right) + \mathrm{Arccosh}\left(\frac{1}{2}\mathrm{Tr}\,\gamma^R\right)}. \tag{2.1}$$

Here γ^L, γ^R are the left and right parts of an isometry $\gamma \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. For example, one makes the wormhole of Fig. 2 rotating by taking two generators γ_1, γ_2 whose left and right parts are not equal. A generator of isometries of the single asymptotic region of this wormhole is then the commutator: $\gamma = [\gamma_1, \gamma_2]$. Substituting its left and right parts into the formula (2.1) one gets the horizon angular velocity. Similarly one can compute the angular velocity of any horizon for a BH of a general type.

We also note that there is a similar formula for the horizon size:

$$2\pi r_+ = \mathrm{Arccosh}\left(\frac{1}{2}\mathrm{Tr}\,\gamma^L\right) + \mathrm{Arccosh}\left(\frac{1}{2}\mathrm{Tr}\,\gamma^R\right). \tag{2.2}$$

Here γ^L, γ^R are also the left and right parts of a generator of isometries of the asymptotic region whose horizon size is being computed. Formulas (2.1), (2.2) allow us to calculate all of the horizon properties in terms of the corresponding group elements. An analytically continued version of these formulas will play an important role in what follows.

3 Analytic Continuation

The content of this section is from [1, 2].

Let us now turn to the procedure of analytic continuation. We first describe this procedure for a non-rotating BH. The basic idea is, instead of analytically continuing the metric in some

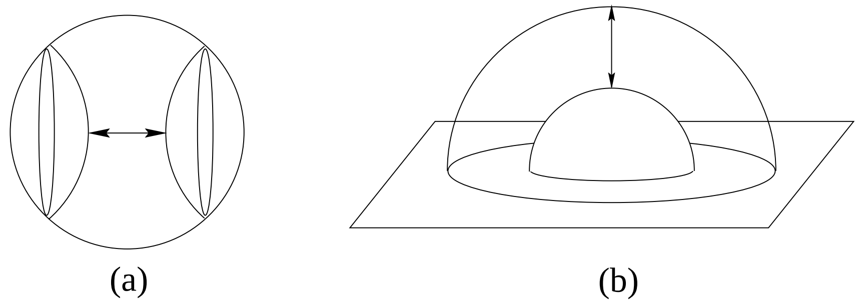


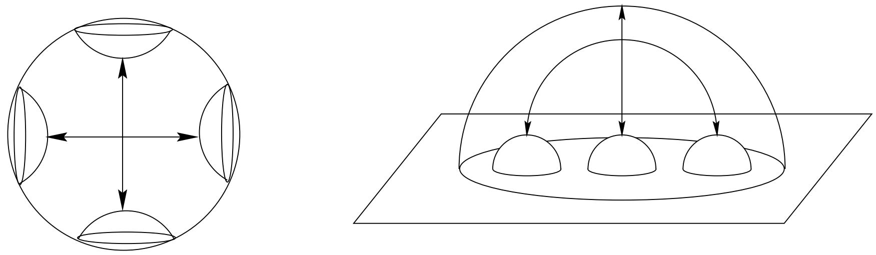
Figure 4: Euclidean BTZ black hole

time coordinate, produce a space $\mathcal{M}$ by identifying points in the Euclidean AdS_3 *using the same group* Γ. That is $\mathcal{M} = \mathbf{H}^3/\Gamma$. We recall that the group of isometries of the Euclidean AdS_3 (=hyperbolic space $\mathbf{H}^3$) is $\mathrm{SL}(2,\mathbb{C})$. However, $\mathrm{SL}(2,\mathbb{R})$ is naturally a subgroup of $\mathrm{SL}(2,\mathbb{C})$, thus Γ acts on $\mathbf{H}^3$ and this action can be used to obtain a quotient space. To see what this quotient space is let us give another, equivalent description of the continuation map. Let us take a section of the Poincare ball (a model for $\mathbf{H}^3$) by a plane passing through the center of the ball. The intersection of the ball with the plane is a unit disk $\mathbf{U}$. Let us call this $t = 0$ plane and do on it the same identifications as we do on the time symmetry plane of the spacetime to be analytically continued. Let us then "evolve" these identifications, but now in the Euclidean time. To do this one just constructs geodesic surfaces intersecting the $t = 0$ plane orthogonally along the geodesics bounding the fundamental region. The geodesic surfaces in $\mathbf{H}^3$ are hemispheres; they are to be identified.

Let us see how this works for the simplest case of the BTZ black hole. Thus, we require that the geometry of the $t = 0$ slice of the unit ball is the same as the geometry of the $t = 0$ slice of BTZ black hole, see Fig. 1. We then have to build geodesic surfaces above and below the two geodesics on the $t = 0$ plane, see Fig. 4(a). The Euclidean BTZ black hole is then the region between these hemispheres; the hemispheres themselves are identified. It is clear that the space obtained is a solid torus, its conformal boundary being a torus. It is often more convenient to work with another model for the same space, that using the upper half space. The interior of the Poincare ball can be isometrically mapped into the upper half-space. The boundary sphere goes under this map into the $x - y$ plane. In the case of Γ generated by a single generator one can always put its fixed points to $0, \infty$, so that the picture of the Euclidean BTZ BH becomes that in Fig. 4(b). It is important that using our procedure we have arrived at the same space as is the one obtained by the usual analytic continuation in the time coordinate, see [8].

Let us consider another example. We now want to construct the Euclidean version of the single asymptotic region black hole with a torus inside the horizon. The procedure is the same: we require a slice of the unit ball to have the same geometry as the $t = 0$ slice of the black hole. This gives us four hemispheres inside the unit ball; the Euclidean space is the region between them and they are to be identified cross-wise, see Fig. 5(a). One sees that the Euclidean space is a solid 2-handled sphere. One can again map the whole configuration into the upper half-space, see Fig. 5(b).

One can do a similar analysis for the three asymptotic region black hole (one also gets a solid two-handled sphere), and for any other of the non-rotating black holes of [4]. In all cases the



(a) (b)

Figure 5: Euclidean single asymptotic region black hole with a torus inside

pattern is the same: one requires the $t = 0$ slice geometry to be the same also in the Euclidean case, and this determines the Euclidean geometry completely. The Euclidean spaces one gets are handlebodies.

We would now like to understand in more detail a relation between the geometry of the time symmetry plane and the conformal boundary $\partial\mathcal{M}$ of the Euclidean space. As we shall explain, the later is the so-called Schottky double of the former. Given a Riemann surface X, closed or with a boundary, its Schottky double is another Riemann surface $\tilde{X}$, not necessarily connected, out of which the original surface can be obtained by identifications: $X = \tilde{X}/\sigma$. Here σ is an anti-holomorphic map of $\tilde{X}$ into itself. For a surface X without a boundary the Schottky double $\tilde{X}$ is given by two disconnected copies of X, with orientation of the second copy reversed. For a surface with a boundary one takes two copies of X and glues them along the boundary to obtain a connected surface. The anti-holomorphic map σ fixes the pre-image of the boundary of X on $\tilde{X}$.

It is not hard to convince oneself that the geometry of the boundary $\partial\mathcal{M}$ of the Euclidean space $\mathcal{M} = \mathbf{H}^3/\Gamma$ obtained via our analytic continuation prescription is that of the Schottky double of the $t = 0$ geometry X. This is related to the fact that the Riemann surface one obtains is uniformized by the complex plane, which is the so-called uniformization by Schottky groups. Let us first recall some basic information about the Schottky groups, see, e.g., [9] as a reference. A Schottky group Σ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$, freely (that is, no relations) generated by a number g of loxodromic (that is $\mathrm{Tr}(L_i) \notin [0,2]$) generators $L_1, \ldots, L_g \in \mathrm{SL}(2,\mathbb{C})$. The Schottky group Σ acts by conformal transformations on the complex plane $\mathbb{C}$. Let us denote by $\mathcal{C}$ the complement of the set of fixed points of this action. As is not hard to convince oneself, the quotient $\mathcal{C}/\Sigma$ is a compact genus g Riemann surface. A Riemann surface obtained from the complex plane by identifications from a Schottky group is called uniformized via Schottky. This is a uniformization different from the usual Fuchsian one that uses the hyperbolic plane. We have already encountered surfaces uniformized by Schottky groups. The boundaries of our Euclidean spaces were obtained exactly this way. It is only that we considered Schottky groups that are real, that is, subgroups of $\mathrm{SL}(2,\mathbb{R})$. This is related to the fact that we have so far only considered non-rotating spacetimes. As we explain later, inclusion of rotation would amount to considering general Schottky groups.

Let us illustrate all this on an example. Consider the single asymptotic region wormhole. The $t = 0$ plane geometry is obtained as a quotient with respect to a group $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ generated by two elements. The same group, thought of as a subgroup of $\mathrm{SL}(2,\mathbb{C})$ acts in $\mathbf{H}^3$. In particular,

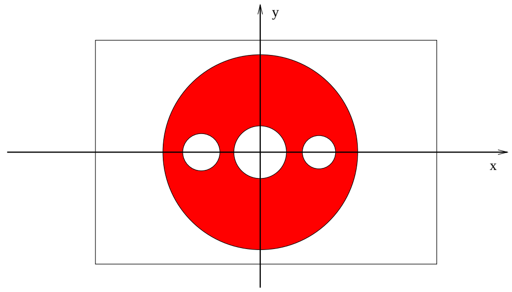


Figure 6: The fundamental domain for the Schottky uniformization of a genus 2 surface. The complex plane here is the conformal boundary of the hyperbolic space. Adding above the plane 4 hemi-spheres based on these circles one gets the configuration depicted in Fig. 5.

it acts on the boundary of $\mathbf{H}^3$, that is the complex plane, by fractional linear transformations. The boundary of the Euclidean space is then obtained as the quotient of the complex plane by this action. The fundamental region for this action is shown in Fig. 6. Since all generators are in $\mathrm{SL}(2,\mathbb{R})$, their fixed points are located on the real axes, and so are the centers of the circles bounding the fundamental region. Removing the circles one gets a sphere with four holes. Identifying their boundaries one gets a genus 2 surface –our Euclidean boundary. Extending the identifications into the bulk of the hyperbolic space one gets the situation depicted in Fig. 5. The $t = 0$ plane geometry is embedded into this figure as the $z - x$ plane, that is as a plane orthogonal to the boundary of $\mathbf{H}^3$ and intersecting it along the real axis. Identifications induced on this $z - x$ plane are precisely those needed to get X. It is now clear that the geometry of the Euclidean boundary $\partial\mathcal{M}$ is precisely that of the double of X. Let us note that the two copies of X needed to obtain the Euclidean boundary are exactly the "same" (but have opposite orientation). Indeed, the configuration of circles in Fig. 6 is invariant under the reflection on the real line. This reflection is exactly the anti-holomorphic map σ that is part of the definition of the Schottky double.

Having explained why the boundary of the Euclidean space is the Schottky double of the initial slice geometry, let use this fact to obtain a simple relation between the number of asymptotic regions h, the number of handles g behind the horizon, and the genus of the Euclidean boundary G. As it is easy to see:

$$G = 2g + h - 1. \tag{3.1}$$

Let us now describe the spaces obtained as the analytic continuation of rotating BH spacetimes. The main idea is [2] to analytically continue the identifications in AdS_3 into isometries of the hyperbolic space. This is achieved by analytically continuing the coordinates of the fixed points of the Lorentzian isometries on the boundary cylinder. The result of this analytic continuation is a complex group $\Sigma \in \mathrm{SL}(2,\mathbb{C})$, so that the Euclidean space that corresponds to a rotating BH spacetime is $\mathcal{M} = \mathbf{H}^3/\Sigma$.

To understand what class of groups Σ may arise we need to understand the meaning of the deformation due to rotation. As is explained in detail in [2], the complex group Σ is best thought of as a certain deformation of the group of the non-rotating spacetime $\Gamma_{\mathrm{non-rot}} \in \mathrm{SL}(2,\mathbb{R})$. The deformation in question is the so-called quasi-conformal deformation due to a Fenchel-Nielsen twist.

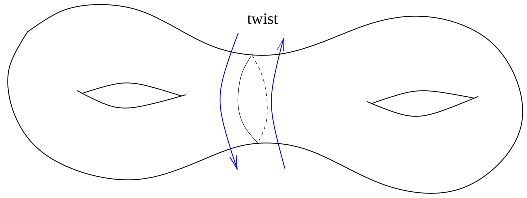


Figure 7: Turning on rotation in some asymptotic region is equivalent to making a Fenchel-Nielsen twist on the corresponding geodesic on $\partial\mathcal{M}$. An example of the single asymptotic region torus wormhole is shown.

The Fenchel-Nielsen twist is a way to change the conformal structure on a Riemann surface. One selects a simple geodesic, cuts the surface along this geodesic, rotates the sides of the cut with respect to each other, and glues back. One obtains a Riemann surface with a different conformal structure. One can describe this new conformal structure either by Fuchsian or quasi-Fuchsian groups. The quasi-Fuchsian description was first given by Wolpert [10]. In this description one starts with a Fuchsian group Γ corresponding to the original surface. The twist along some geodesic corresponds to a particular quasi-conformal f^τ map of the upper half-plane into the interior of a Jordan curve. The image of the real axis is the Jordan curve itself. The resulting deformed group $\Gamma^\tau = f^\tau \circ \Gamma \circ (f^\tau)^{-1}$ is a quasi-Fuchsian group, subgroup of $\mathrm{SL}(2,\mathbb{C})$.

As was explained in [2] the description in terms of quasi-Fuchsian groups is the one relevant in our case. One starts with a non-rotating spacetime. Its analytic continuation is the manifold $\mathcal{M} = \mathbf{H}^3/\Gamma$, whose boundary $\partial\mathcal{M}$ is a Riemann surface: the Schottky double of the time symmetry plane. There is a set of simple closed geodesics on $\partial\mathcal{M}$ that correspond to asymptotic regions (horizons). Turning on rotation in one of the asymptotic regions is equivalent to making a Fenchel-Nielsen twist along the corresponding horizon geodesic on $\partial\mathcal{M}$, see Fig. 7. As the result of the twist one gets a quasi-Fuchsian group $\Sigma \in \mathrm{SL}(2,\mathbb{C})$. The size and angular velocity of all horizons can be easily expressed in terms of traces of certain group elements from Σ. Taking any group element $A \in \Sigma$ whose axis projects on a horizon geodesics on $\partial\mathcal{M}$ one has:

$$\frac{1}{2}\mathrm{Tr}A = \cosh\left(\frac{\pi(r_+ + i|r_-|)}{l}\right), \tag{3.2}$$

where r_+, r_- are the outer and inner horizon radii correspondingly. The mass and angular momentum of the horizon can be expressed in terms of r_+, r_- by simple formulas, see below. Thus, the analytic continuation of the rotating spacetime is the manifold $\mathcal{M} = \mathbf{H}^3/\Sigma$, where Σ is a complex group. The moduli of the Riemann surface $\partial\mathcal{M}$, such as (3.2) describe physical properties of the BH.

Let us now discuss what is the most general group $\Sigma \in \mathrm{SL}(2,\mathbb{C})$ that corresponds to a BH spacetime. The boundary $\partial\mathcal{M}$ of the analytically continued BH spacetime is a quotient $\mathcal{C}/\Sigma$, where $\mathcal{C}$ is the domain of discontinuity for the action of Σ on $\mathbb{C}$. The surface $\partial\mathcal{M}$ is the double of a Riemann surface X with g handles and h holes. The number of moduli describing the conformal geometry of X is $6g + 3h - 6$, out of which h moduli describe the sizes of the holes. The most general $\partial\mathcal{M}$ is obtained by taking two surfaces X with different conformal geometry, and gluing them across the holes. The only condition is that the sizes of holes in the two copies of X must be the same. However, there is an additional parameter –twist– for each hole. Thus, overall, one gets $2(6g+3h-6)$ real parameters. This equals $6G-6$, the dimension of the moduli space at genus G, where G is the genus of $\partial\mathcal{M}$. Thus, the dimension of the moduli space of Σ that correspond to BH spacetimes is just the dimension of the Teichmüller space at genus G.

To describe such Σ's more explicitly we need a version of the Bers' simultaneous uniformization theorem. In its classical version this theorem states that, given two Riemann surfaces X, Y of the same genus and opposite orientation, there exists a quasi-Fuchsian group $\Sigma \in \mathrm{SL}(2,\mathbb{C})$ that simultaneously uniformizes them. In other words the surface $\mathcal{C}/\Sigma$, where as usual $\mathcal{C}$ is the domain of discontinuity for the action of Σ on $\mathbb{C}$, is the disconnected sum $X \cup Y$. The theorem is proved by showing that there exist a quasi-conformal map f^1, which is conformal in the lower half-plane, such that $f^1 \circ \Sigma \circ (f^1)^{-1}$ equals to the Fuchsian group Γ_X uniformizing X. Similarly, there exists a quasi-conformal map f^2, conformal in the upper half-plane, such that $f^2 \circ \Sigma \circ (f^2)^{-1}$ is the Fuchsian group Γ_Y uniformizing Y.

What we need is a version of this theorem for simultaneous uniformization of two surfaces with holes. Let X, Y be two surfaces of type (g,h) of opposite orientation. The statement that we need is that given X, Y and a set of h numbers –twists– that tell us how to glue X to Y across the holes, there exists a complex group Σ that uniformizes the surface obtained by gluing X to Y. Similarly to the compact X, Y case one expects that there are two quasi-conformal maps, conformal in the lower and upper half-planes correspondingly, such that Σ "untwisted" by these maps gives the Fuchsian groups Γ_X, Γ_Y of the second kind uniformizing X, Y. This would be a natural analog of the classical simultaneous uniformization. Unfortunately, we are not aware of a statement to this effect in the literature. It would be of interest to generalize simultaneous uniformization to the case of surfaces with holes.

The groups Σ that arise as a result of such simultaneous uniformization of two surfaces with holes X, Y are the most general groups that correspond to BH spacetimes. The physical properties of BH's can be read off from the group directly using the formula (3.2). The groups arising are finitely generated, free, purely loxodromic Kleinian groups with one component. According to a theorem due to Maskit [11] these are exactly the Schottky groups. Thus, our Σ are Schottky groups, and the moduli space of analytic continuations of BH spacetimes is the Schottky space. We are now ready to calculate the thermodynamic partition function for our BH's. We do this by evaluating the Einstein-Hilbert action on the Euclidean 3-manifolds $\mathcal{M}$ –analytic continuation of BH spacetimes. The partition function will be a certain function on the Schottky space.

4 Partition Function

This section is based on [1, 12]. The Einstein-Hilbert action to be evaluated on spaces $\mathcal{M}$ is given by:

$$I[g] = -\frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{g}(R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \sqrt{\gamma}K + \frac{l}{8\pi G}\int_{\partial\mathcal{M}} \sqrt{\gamma}. \tag{4.1}$$

Here γ is the restriction of the metric g on $\mathcal{M}$ to the boundary $\partial\mathcal{M}$. The last boundary term proportional to the boundary area is necessary to regularize the action. To evaluate the action on $\mathcal{M} = \mathbf{H}^3/\Sigma$ one must select a regularizing family of surfaces. There is a canonical family in $\mathbf{H}^3$ with the following properties: (i) it is compatible with identifications one makes to get $\mathcal{M}$; (ii) each surface has constant negative curvature. Explicitly, the surfaces are given by:

$$\begin{aligned} \xi &= \frac{\rho\, e^{-\varphi/2}}{1+\frac{1}{4}\rho^2 e^{-\varphi}|\varphi_w|^2}, \\ y &= w + \frac{\varphi_{\bar w}}{2}\frac{\rho^2 e^{-\varphi}}{1+\frac{1}{4}\rho^2 e^{-\varphi}|\varphi_w|^2}. \end{aligned} \tag{4.2}$$

Here ξ, y are the usual coordinates in $\mathbf{H}^3$ in the upper half-space model. Fixing $\rho = const$ one gets surfaces in question. Coordinates $w, \bar w$ (and ρ) are Gaussian coordinates based on these surfaces.

The key quantity in (4.2) is the *canonical Liouville field* φ. It is a (real) function of the complex coordinate $w \in \mathcal{C} : \varphi = \varphi(w, \bar w)$. It depends in a certain way on the Schottky group Σ: it satisfies the Liouville equation on the Schottky domain $\mathcal{C}$ (domain of discontinuity of the action of Σ on $\mathbb{C}$) and has the following transformation property:

$$\varphi(Lw) = \varphi(w) - \ln|L'|^2. \tag{4.3}$$

The Liouville field can be constructed if the map between the Schottky and Fuchsian uniformization domains is known, see, e.g., [13] for more details.

Evaluation of (4.1) on $\mathcal{M}$ reduces to computation of the volume of the part of the fundamental region in $\mathbf{H}^3$ that lies above a surface $\rho = \epsilon$. One then subtracts a multiple of the area of this surface, removes a simple logarithmic divergence proportional to the Euler characteristic $\chi = 2 - 2G$, and takes the limit $\epsilon \to 0$. The result of this computation is the Liouville action on $\partial\mathcal{M}$, as defined in [13], evaluated on the canonical Liouville field φ:

$$I[\mathcal{M}] = -\frac{l}{4G} I_{\mathrm{Liouv}}[\varphi]. \tag{4.4}$$

See [1, 12] for more details on this computation. As it was shown in [13], the on-shell Liouville action appearing in (4.4) is the Kähler potential for the Weil-Peterson metric on the Schottky space. In other words, the (logarithm of the) thermodynamic partition function of a BH, as a function of BH physical parameters (encoded by moduli of $\partial\mathcal{M}$) is the Kähler potential on the BH moduli space. Knowing the BH partition function we are ready to study the BH thermodynamics.

5 Thermodynamics

The content of this section is new. Our goal will be to obtain the BH entropy. We use the standard thermodynamics relations. What we need is analogs of relations (1.9), (1.10). It is instructive to consider the case of the BTZ BH first. Our treatment of the BTZ BH is reminiscent of [8].

Let us first introduce convenient notations. Define:

$$\alpha = \frac{p}{2b}, \qquad p = \frac{r_+ + i|r_-|}{l}, \tag{5.1}$$

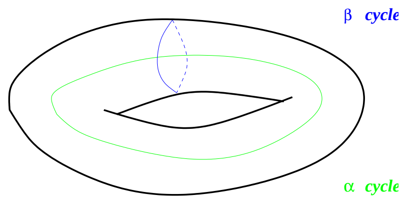


Figure 8: The alpha- and beta-cycles on the boundary of the Euclidean BTZ BH. The alpha-cycle is non-contractible inside the hole. The beta-cycle is contractible, and is in the Euclidean time direction.

and b is defined via $1/b^2 = l/4G$. We remind the reader that l is the radius of curvature of AdS, defined as $l = 1/\sqrt{-\Lambda}$, where Λ is the cosmological constant. We set $\hbar = 1$, so that $1/b^2$ is essentially the ratio of radius of curvature of AdS to the Planck length. Semi-classical regime of small curvatures corresponds to $b \to 0$. Define:

$$\Delta - \Delta_0 = \alpha^2. \tag{5.2}$$

All these quantities have interpretation in terms of Liouville theory: b is the Liouville coupling constant and $\Delta - \Delta_0$ is the conformal dimension of a "primary state" α, minus the conformal dimensions of the lowest lying state. This relation to Liouville theory won't play any role in this paper, we mention it only to provide an explanation for the relations above. The mass and angular momentum of the hole can now be expressed entirely in terms of the quantity α, more precisely in terms of the conformal dimension. One has:

$$\begin{aligned} Ml &= (\Delta - \Delta_0) + \overline{(\Delta - \Delta_0)}, \\ J/8G &= (\Delta - \Delta_0) - \overline{(\Delta - \Delta_0)}. \end{aligned} \tag{5.3}$$

Here M, J are the BH mass and angular momentum. Thus, we see that M, J are completely determined by the parameter α. It will be much more convenient to use the complex parameter α as a physical parameter describing the hole.

To introduce an analog of the relation (1.9) we, following [8], consider an "off-shell" BTZ BH. In the Euclidean BTZ BH the alpa-cycle on the boundary is not-contractible inside the space, see Fig. 8. The corresponding $\mathrm{SL}(2,\mathbb{C})$ holonomy is non-trivial and is just the generator of $\Sigma \subset \mathrm{SL}(2,\mathbb{C})$, the group that we use to obtain the space $\mathcal{M} = \mathbf{H}^3/\Sigma$. The other, conjugate generator of $\pi_1(\partial\mathcal{M})$, the beta-cycle is contractible inside $\mathcal{M}$. The corresponding $\mathrm{SL}(2,\mathbb{C})$ holonomy is trivial. The off-shell BTZ BH is obtained by allowing the beta-cycle holonomy to be non-trivial. Thus, one takes:

$$\frac{1}{2}\mathrm{Tr}A_\beta = \cos\pi(\phi + i\chi), \tag{5.4}$$

and introduces a variable:

$$\beta = \frac{1}{2b}(\phi + i\chi). \tag{5.5}$$

The on-shell BTZ BH is obtained by setting $\phi = 1, \chi = 0$, in other words $\beta = 1/2b$.

We should now calculate the partition function for the off-shell BTZ BH. It is a straightforward exercise to do this. First, we note that for the on-shell BTZ the Euclidean action one gets is:

$$I_{\rm BTZ} = -\frac{2\pi r_+}{8G}. \tag{5.6}$$

It equals to minus half of the corresponding Bekenstein-Hawking entropy. This result can be obtained by evaluating the volume between two hemi-spheres in $\mathbf{H}^3$, regularized by the family of surfaces (4.2), which in the BTZ case is simply a family of cones with the tip at the origin. See, e.g., [1] for this calculation. Modification for an off-shell BTZ is simple. Note that when $\phi \neq 1, \chi = 0$, one has to cut out a wedge of angle $2\pi\phi$ out of $\mathbf{H}^3$. This means that the factor of 2π in (5.6) will be replaced by $2\pi\phi$. One can similarly calculate the action for the general case $\chi \neq 0$. The result when written in terms of α, β complex variables is:

$$\ln Z = 2\pi \left(\alpha\beta + \overline{\alpha\beta}\right). \tag{5.7}$$

We wrote the result in terms of the partition function: $\ln Z = -I$. Thus, we have the following relation between the α and β parameters:

$$\alpha = \frac{1}{2\pi}\frac{\partial \ln Z}{\partial \beta}. \tag{5.8}$$

This is the desired analog of the relation (1.9) for BTZ BH. Of course, one could have also take a derivative of the $\ln Z$ with respect to the usual inverse temperature, as it was done in (1.9). What one would obtain is the BTZ mass (5.3). It is, however, more convenient to work with the parameters α, β directly related to the holonomies of the BH.

The entropy can now be obtained using:

$$S = \left(\beta\frac{\partial \ln Z}{\partial \beta} + \bar\beta\frac{\partial \ln Z}{\partial \bar\beta}\right) + \ln Z = 2\pi\left(\alpha\beta + \overline{\alpha\beta}\right) + \ln Z. \tag{5.9}$$

Here we have used (5.8) to write the second equality. To obtain the on-shell BTZ BH entropy we must set $\beta = 1/2b$ in the above expression. One obtains:

$$S_{\rm BTZ} = \frac{2\pi r_+}{4G}. \tag{5.10}$$

This is the Bekenstein-Hawking entropy equal to the horizon length devided by $4G$.

Using this example as a guide, let us now consider a general BH of type (g, h). Its analytic continuation is the 3-manifold $\mathcal{M}$, whose boundary $\partial\mathcal{M}$ is a Riemann surface of genus $G = 2g+h-1$. The boundary is marked with a set of h simple closed geodesics: horizon geodesics. We shall refer to these geodesics as alpha-cycles. In addition, there is a set of beta-cycles on $\partial\mathcal{M}$. These are such that the $\mathrm{SL}(2,\mathbb{C})$ holonomy along them is trivial. They are contractible inside $\mathcal{M}$. Each beta-cycle is intersected by exactly two alpha-cycles, see Fig. 9 for a drawing of the cycles for a 3-asymptotic region BH.

The first step towards general BH thermodynamics would be to evaluate the partition function for a general off-shell BH, in which both alpha- and beta-cycles are non-trivial. This is equivalent to a problem of evaluating the action (4.1) on a 3-manifold with a set of conical singularities inside.

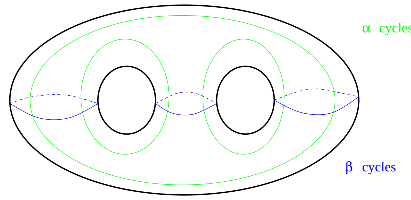


Figure 9: Alpha- and beta-cycles on the boundary of the Euclidean 3-asymptotic region BH.

The moduli space of such manifolds is parameterized by homomorphisms of $\pi_1(\partial\mathcal{M})$ into $\mathrm{SL}(2,\mathbb{C})$, modulo conjugations by $\mathrm{SL}(2,\mathbb{C})$. This moduli space has the real dimension $12G-12$. The on-shell Euclidean BH's form a submanifold of real dimension $6G-6$ in it. The moduli space of off-shell BH's is naturally a symplectic manifold. The symplectic form is that of CS theory on $\partial\mathcal{M}$. Some properties of this phase space were studied in [14], where also a relation to the most general solution of 2+1 gravity with negative cosmological constant was established. However, those details will not be of importance for us here.

Let us denote by $I_{\mathrm{off-shell}}$ the result of the evaluation of the action (4.1) on an off-shell BH. Introducing the parameters α, β for all alpha- and beta-cycles, the off shell action becomes a function of these parameters. The on-shell result (4.4) suggests that the general off-shell action is the Kähler potential on the general moduli space

$$\mathrm{Hom}\left(\pi_1(\partial\mathcal{M}), \mathrm{SL}(2,\mathbb{C})\right)/\mathrm{SL}(2,\mathbb{C}). \tag{5.11}$$

As α, β are the canonically conjugate quantities on this moduli space, this would imply that

$$\alpha = \frac{1}{2\pi}\frac{\partial \ln Z}{\partial \beta}, \tag{5.12}$$

exactly like what we had in the BTZ BH case. Note, however, that the function $\ln Z$ in the general case is much more complicated and is not given by a simple product of $\alpha\beta$ like in the BTZ case. In fact, it is an outstanding problem to compute $\ln Z$ in some explicit fashion.

Assuming that our conjecture about the off-shell $\ln Z$ being the Kähler potential on (5.11) is correct, we can compute the BH entropy. It is given by an analog of the formula (5.9):

$$S = \sum \left(\beta\frac{\partial \ln Z}{\partial \beta} + \bar{\beta}\frac{\partial \ln Z}{\partial \bar{\beta}}\right) + \ln Z = 2\pi\left(\alpha\beta + \overline{\alpha\beta}\right) + \ln Z. \tag{5.13}$$

Here the sum is taken over all the horizon geodesics, and we have used (5.12) to write the second equality. We are interested in the on-shell BH entropy. To get this, we need to set all $\beta = 1/2b$ in (5.13). We get:

$$S = \sum \frac{2\pi r_+}{8G} + \ln Z. \tag{5.14}$$

Thus, the on-shell entropy is given by the sum of horizon circumferences over $8G$ plus the logarithm of the partition function, proportional to the Kähler potential on the Schottky space. This is our main formula for the entropy of a general BH. We see that the entropy is not given by a simple sum of length of the horizons. There is a contribution depending non-trivially on the other moduli. It would be of interest to study this dependence in detail.

We could have derived (5.14) without any recourse to the off-shell BH. Indeed, we could have written for the entropy:

$$S = \sum (\beta M + \Phi J) + \ln Z, \tag{5.15}$$

where the sum is taken over all horizons, and M, J are the usual mass and angular momentum in an asymptotic region, and β, Φ are the thermodynamically conjugate quantities. Since every asymptotic region is indistinguishable from that of the BTZ BH, the quantity in brackets in (5.15) is equal to $2\pi r_+/8G$, which gives (5.14).

Let us now note that physical considerations, namely the Bekenstein bound for the entropy of a system, suggest that the entropy of a general BH of type (g, h) is bounded by the sum of horizon circumferences over $4G$:

$$S \leq \sum \frac{2\pi r_+}{4G}. \tag{5.16}$$

In view of (5.14) the bound (5.16) becomes a bound on the Kähler potential that can be written as:

$$I_{\text{Liouv}}[\Sigma] \leq \sum \log |m|. \tag{5.17}$$

Here $I_{\text{Liouv}}[\Sigma]$ is the Kähler potential on the Schottky space that appears in (4.4), and is equal to the Liouville action evaluated on the canonical Liouville field; Σ is a point in the Schottky space. The sum is taken over all horizon geodesics, and m is the mutliplier of the corresponding transformation, that is the modulus greater than 1 eigenvalue of the corresponding $\text{SL}(2, \mathbb{C})$ matrix. The formula (5.14), together with the bound (5.17) that it suggests for the Kähler potential, is our main result.

6 Summary

We have obtained that the entropy of a general BH of type (g, h) is given by (5.14). In words, the entropy is given by the sum of half of Bekenstein-Hawking entropies for every horizon, plus the logarithm of the partition function. In the case of the BTZ BH the logarithm of the partition function equals to the half of the Bekenstein-Hawking entropy, which gives the entropy equal to the Bekenstein-Hawking value. For a general BH the entropy is not given by a simple sum of horizon circumferences. Indeed, the function $\ln Z$, which has the meaning of the Kähler potential on the Schottky space (moduli space of BH's) depends non-trivially on the geometry of the BH interior. At best, the entropy can be expected to be bounded from above by the Bekenstein-Hawking value, which leads us to conjecture a very interesting bound (5.17) on the Kähler potential. It would be of considerable interest to establish this bound by pure mathematical means, without any reference to BH physics. It would also be of interest to find if (and when) the bound can be saturated. In other words, are there any other BH's apart from the BTZ whose entropy is equal to the sum of horizon circumferences over $4G$? We leave these interesting questions to future work.

We conclude with a brief discussion of the physical interpretation of the entropy (5.14) we obtained. Let us first note that the analytic continuation procedure that we used was non-standard.

So is the entropy result (5.14) that we obtained. Indeed, instead of analytically continuing the discrete group of identifications, we could have continued the time coordinate. Even though there is no global KVF whose affine parameter can serve as a time coordinate covering the whole spacetime (such a KVF exists only in the BTZ BH case), there is such a coordinate in every asymptotic region. Indeed, every such region is indistinguishable from that of the BTZ BH. Thus, one can analytically continue the metric in some asymptotic region. The resulting space would be a solid torus, whose modular parameter is determined by the horizon size of the continued asymptotic region. The entropy in *that* asymptotic region would be equal to the corresponding horizon circumference over $4G$. One can argue that this is the entropy that is "observable" by an observer that lives in that asymptotic region. In contrast, the entropy obtained using our analytic continuation procedure is that associated to the whole spacetime, not just a single horizon or a single asymptotic region. As such it is not a quantity that can be detected or measured by an observer in a single asymptotic region. Instead, this is what a *super-observer* that knows about the existence of all asymptotic regions (and about the topology inside the BH) would call the BH entropy. The entropy (5.14) is that for a BH of a particular internal geometry, and it depends on this geometry non-trivially.

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